Quantum Cosmology of de Sitter space: Wheeler-DeWitt and Path Integral Approaches

Lorenz Hartmann

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Outline

Cosmological Model (de Sitter Space) + Principles of Quantum Mechanics

Canonical Quantization, Wheeler-DeWitt equation [1]

Path Integral [6,7]

Wave function Ψ $\xleftrightarrow{?}$ Wave function Ψ

Canonical Quantization

- Consider Einstein-Hilbert action

$$S = S_{grav} + S_{boundary}$$
$$= \int_M d^4x\sqrt{-g}\left[\frac{\mathcal{R}}{2} - \Lambda\right] - \int_{\partial M} d^3x\sqrt{h}K$$

$$\Big\downarrow \quad \text{ansatz: (closed) FLRW metric} \quad ds^2 = -N(t)^2dt^2 + a(t)^2d\Sigma^2$$

$$S = 2\pi^2 \int dt \left(-3\frac{\dot{a}^2 a}{N} + 3Na - Na^3\Lambda\right)$$

- Treat this as quantum mechanical system with coordinates *a* and *N*

- The lapse function *N* represents gauge freedom which leads to a constraint equation

$$0 \overset{!}{=} \mathcal{H} = -\frac{p^2}{24\pi^2 a} - 6\pi^2 a + 2\pi^2 a^3 \Lambda$$

where *p* is the momentum conjugate to *a*

[4,5]

- Quantum theory: constraint on physical states

$$\boxed{\hat{\mathcal{H}}\Psi = 0}$$

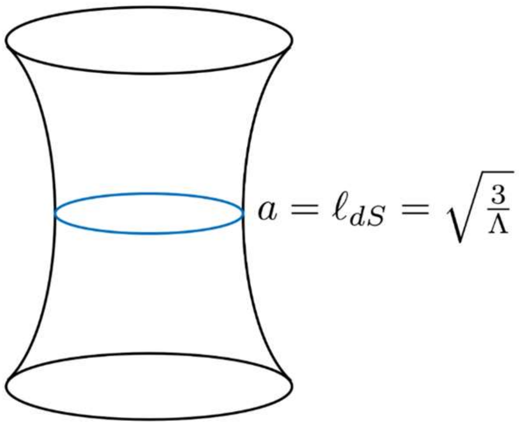
Wheeler-DeWitt (WDW) equation

Solutions to the Wheeler-DeWitt equation

Classical de Sitter space

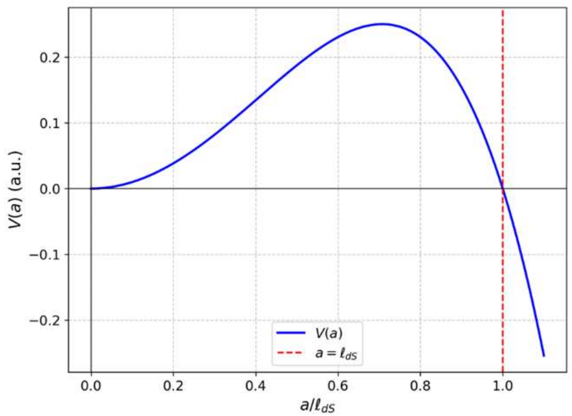
$$ds^2 = -dt^2 + a(t)^2 d\Omega^2$$

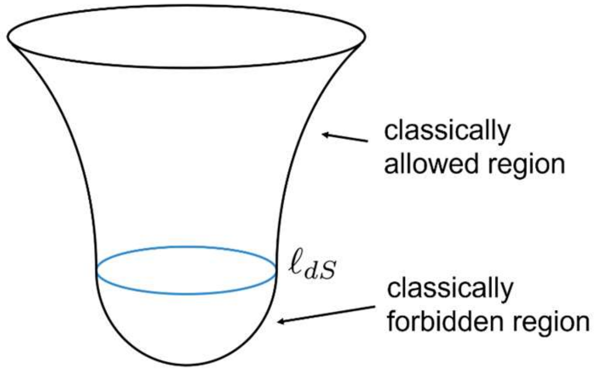
$$a(t) = \ell_{dS} \cosh(t/\ell_{dS}) \quad ; \quad \ell_{dS} = \sqrt{\frac{3}{\Lambda}}$$



Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi \propto \left(-\frac{1}{144\pi^4}\frac{d^2}{da^2} + a^2\left(1 - \frac{\Lambda}{3}a^2\right)\right)\Psi = 0$$





Looks like 'Schrödinger equation' for particle in a potential

[2,3]

- Imagine universe „tunnelling“ through the potential barrier. Wave function picks up a factor of

$$T = \exp\left(\pm \int_0^{\ell_{dS}} da\, \sqrt{144\pi^4\, V(a)}\right) = \exp\left(\pm\frac{12\pi^2}{\Lambda}\right)$$

\+ : Hartle-Hawking (HH) solution
\- : Linde-Vilenkin (LV) solution

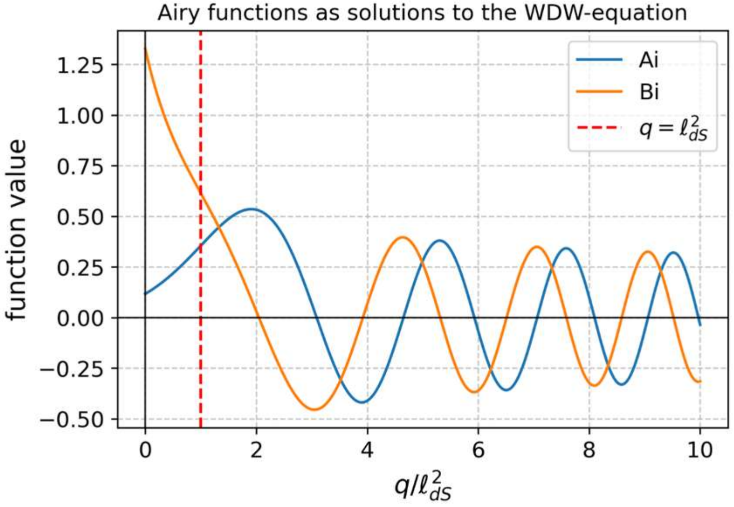
The sign difference is crucial since it tells us which values of the vacuum energy (cosmological constant, scalar potential, ...) are more likely

- Choosing a different operator ordering when quantizing, one can solve WDW equation exactely

$$\boxed{\Psi = c_1 Ai(z) + c_2 Bi(z)}$$

[2,3]

$$z = \frac{(12\pi^4)^{1/3}}{\Lambda^{2/3}}(3 - \Lambda q)\,, \quad q = a^2$$



Path Integral Approach

- Consider the propagator for transitions between initial and final state – setting the initial scale factor to zero gives the wave function

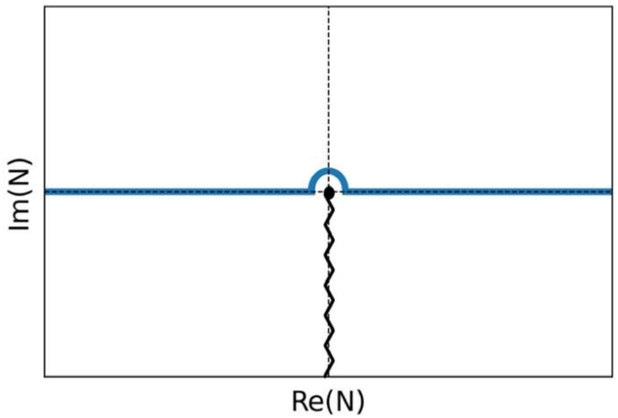
$$G = G(i \to f) = \int_i^f \mathcal{D}g_{\mu\nu}\, e^{iS[g_{\mu\nu}]} \qquad\qquad i = (a_i) \quad \to \quad f = (a_f)$$

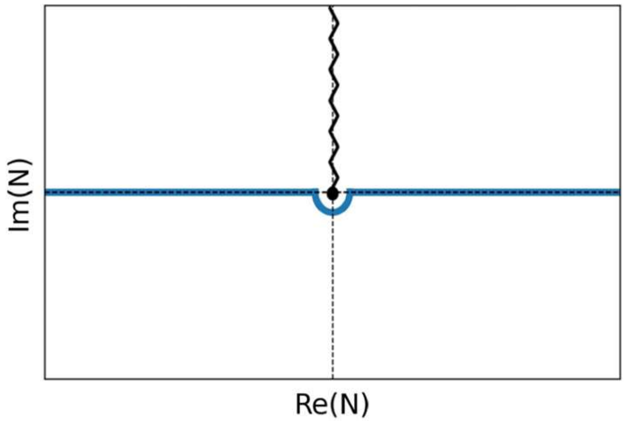
- This integral can be simplified to an ordinary, one-dimensional integral over the lapse [7]

$$G = \sqrt{\frac{3i\pi}{2}} \int_C \frac{dN}{\sqrt{N}} e^{iS_0}$$

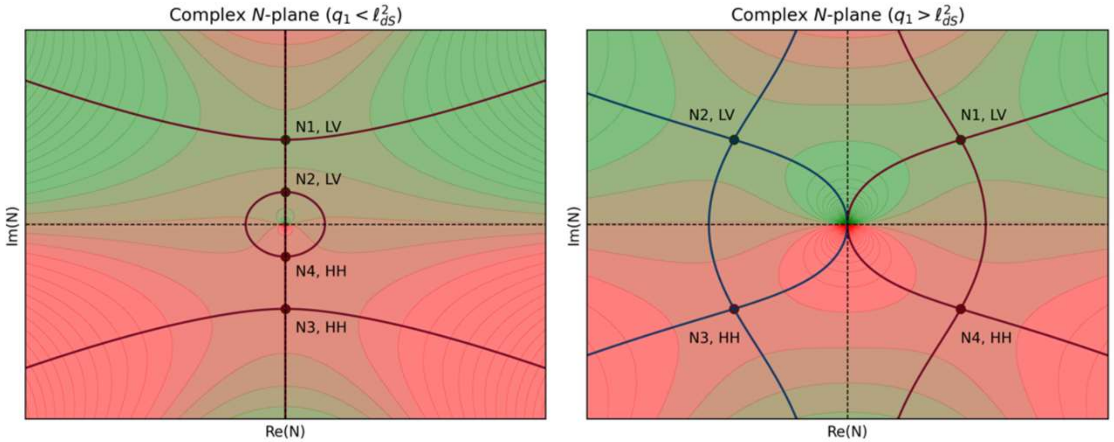
$$S_0 = 2\pi^2 \left(\frac{\Lambda^2 N^3}{36} + N \left(3 - \frac{1}{2}\Lambda(q_0 + q_1) \right) - \frac{3}{4N}(q_1 - q_0)^2 \right)$$

- Connection to Canonical Quantization: One can show that this propagator satisfies the Wheeler-DeWitt equation if one integrates over the whole real lapse line. [9]
  This leads to <u>two possibilites</u>:





- The integrals can be analyzed with Picard-Lefschetz theory, deforming the contours to (complex) steepest-descent contours along which the integrand does not oscillate but decay exponentially [8]



- The integral can be approximated by its behaviour around the saddle points

- Conclusions:
  - Both contour choices lead to a convergent integral
  - Closing the contour from above gives a LV wave function Ψ_{LV}, closing the contour from below gives a HH wave function Ψ_{HH}

Exact Solutions to the Path Integral

- Idea: If the Path Integral is too complicated to solve, evaluate it at special values and ‘match‘ the result to the known solutions of the WDW equation [10]

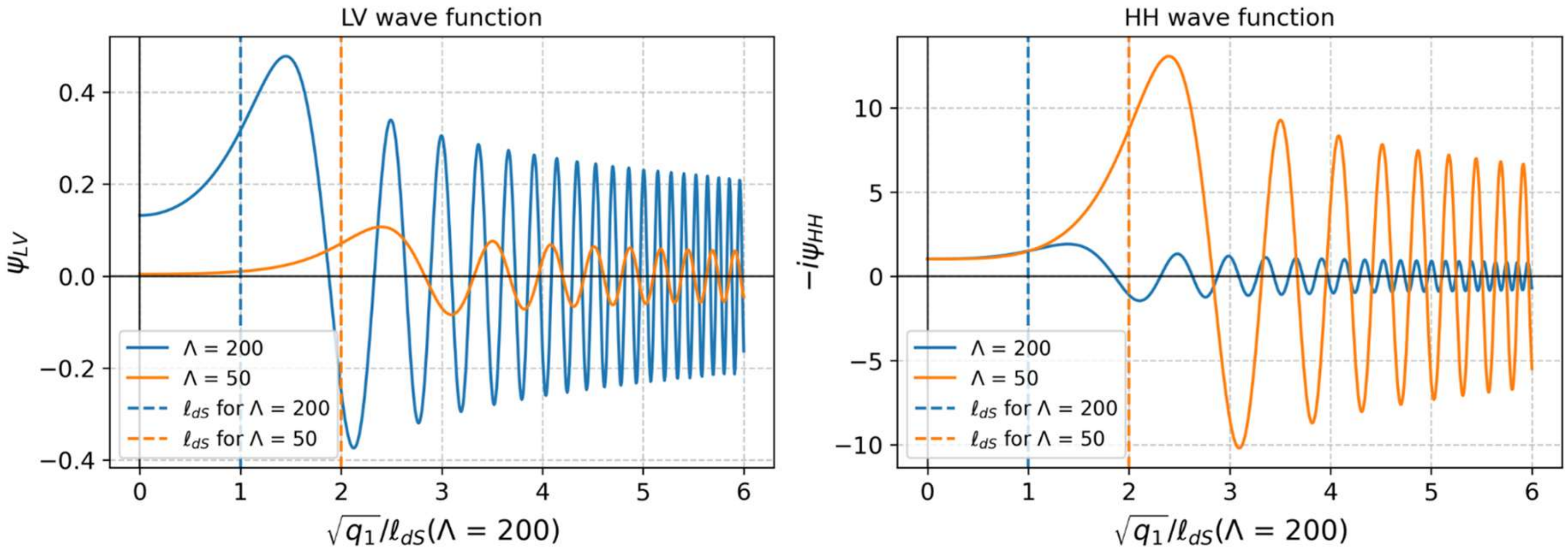
- One finds exact solutions to the path integral in terms of Airy functions

$$\Psi_{LV}(q) = \frac{\pi^{5/3}2^{4/3}3^{2/3}}{\Lambda^{1/3}} Ai(z(0))Ai(z(q))$$

$$\Psi_{HH}(q) = i\frac{\pi^{5/3}2^{1/3}3^{2/3}}{\Lambda^{1/3}} (Ai(z(0))Bi(z(q)) + Ai(z(q))Bi(z(0)))$$

$$z(q) = \frac{(12\pi^4)^{1/3}}{\Lambda^{2/3}}(3 - \Lambda q), \quad q = a^2$$

Both are valid candidates for the wave function of the universe



- Unlike Ψ_{LV} , Ψ_{HH} is <u>independent of Λ</u> for small scale factors. This is an argument in favour of the Hartle-Hawking wave function because the cosmological constant should be negligible for small universes

Summary

- The wave function can be calculated from both the Wheeler-DeWitt equation and the path integral

- With an appropriate choice of integration contour for the lapse, the path integral satisfies the WDW equation

- The wave functions obtained is this way are of LV or HH type, depending on the contour

- While the wave functions both have their advantages, Ψ_{HH} has the unique feature of being independent of Λ for small scale factors

References and Further Reading

- Wheeler-DeWitt equation and its solutions

[1] Bryce S. DeWitt. Quantum theory of gravity. i. the canonical theory. *Phys. Rev.*, 160:1113–1148, Aug 1967.

[2] Alexander Vilenkin. Approaches to quantum cosmology. *Physical Review D*, 50(4):2581–2594, August 1994.

[3] Alexander Vilenkin. Quantum cosmology and the initial state of the universe. *Phys. Rev. D*, 37:888–897, Feb 1988.

- Quantization of constrained systems

[4] P. A. M. Dirac. Generalized hamiltonian dynamics. *Canadian Journal of Mathematics*, 2:129–148, 1950.

[5] Marc Henneaux and Claudio Teitelboim. *Quantization of Gauge Systems*, pages 3–47. Princeton University Press, 1992.

- The Path Integral and Picard-Lefschetz theory

[6] J. B. Hartle and S. W. Hawking. Wave function of the universe. *Phys. Rev. D*, 28:2960–2975, Dec 1983.

[7] Jean-Luc Lehners. Review of the no-boundary wave function. *Physics Reports*, 1022:1–82, June 2023.

[8] Job Feldbrugge, Jean-Luc Lehners, and Neil Turok. Lorentzian quantum cosmology. *Phys. Rev. D*, 95:103508, May 2017.

[9] Jonathan J. Halliwell. Derivation of the wheeler-dewitt equation from a path integral for minisuperspace models. *Phys. Rev. D*, 38:2468–2481, Oct 1988.

[10] Jonathan J. Halliwell and Jorma Louko. Steepest-descent contours in the path-integral approach to quantum cosmology. i. the de sitter minisuperspace model. *Phys. Rev. D*, 39:2206–2215, Apr 1989.